

Discriminantly separable polynomials and quad-graphs

Vladimir Dragović (*) and Katarina Kukić (**)

- (*) The Department of Mathematical Sciences, University of Texas at Dallas,
800 West Campbell Road, Richardson TX 75080, USA
Mathematical Institute SANU, Kneza Mihaila 36, 11000 Belgrade, Serbia
e-mail: Vladimir.Dragovic@utdallas.edu
- (**) Faculty for Traffic and Transport Engineering, University of Belgrade
Vojvode Stepe 305, 11000 Belgrade, Serbia
e-mail: k.mijailovic@sf.bg.ac.rs

Abstract

We classify the discriminantly separable polynomials of degree two in each of three variables, defined by a property that all the discriminants as polynomials of two variables are factorized as products of two polynomials of one variable each. Our classification is based on the study of structures of zeros of a polynomial component P of a discriminant.

From a geometric point of view, such a classification is related to the types of pencils of conics. We construct also discrete integrable systems on quad-graphs associated with the discriminantly separable polynomials. We establish a relationship between our classification and the classification of integrable quad-graphs which has been suggested recently by Adler, Bobenko and Suris. As a fit back benefit, we get a geometric interpretation of their results in terms of pencils of conics. In the case of general position, when all four zeros of the polynomial P are distinct, we get a connection with the Buchstaber-Novikov two-valued groups on \mathbb{CP}^1 .

AMS Subj. Class. 37K60 (14N05, 37K20)

Contents

1	Introduction: an overview on discriminantly separable polynomials	2
2	Classification of strongly discriminantly separable polynomials of degree two in three variables	5
3	From discriminantly separable polynomials to integrable quad-graphs	11

1 Introduction: an overview on discriminantly separable polynomials

In a recent paper [13] of one of the authors of the present paper, a new approach to the Kowalevski integration procedure from [19] has been suggested. It has been based on a new notion introduced therein of *the discriminantly separable polynomials*. A family of such polynomials has been constructed there as pencil equations from the theory of conics

$$\mathcal{F}(w, x_1, x_2) = 0,$$

where w, x_1, x_2 are the pencil parameter and the Darboux coordinates respectively. (For classical applications of the Darboux coordinates see Darboux's book [11], for modern applications see the book [15] and [12].) The key algebraic property of the pencil equation, as quadratic equation in each of three variables w, x_1, x_2 is: *all three of its discriminants are expressed as products of two polynomials in one variable each*:

$$\begin{aligned}\mathcal{D}_w(\mathcal{F})(x_1, x_2) &= P(x_1)P(x_2) \\ \mathcal{D}_{x_1}(\mathcal{F})(w, x_2) &= J(w)P(x_2) \\ \mathcal{D}_{x_2}(\mathcal{F})(w, x_1) &= P(x_1)J(w)\end{aligned}\tag{1}$$

where J, P are polynomials of degree 3 and 4 respectively, and the elliptic curves

$$\Gamma_1 : y^2 = P(x), \quad \Gamma_2 : y^2 = J(s)$$

are isomorphic (see Proposition 1 of [13]) .

The so-called *fundamental Kowalevski equation* (2) (see [19], [18], [17]) appeared to be an example of a member of the family, as it has been observed in [13]:

$$Q(s, x_1, x_2) := (x_1 - x_2)^2 \left(s - \frac{l_1}{2}\right)^2 - R(x_1, x_2) \left(s - \frac{l_1}{2}\right) - \frac{1}{4} R_1(x_1, x_2) = 0, \quad (2)$$

where $R(x_1, x_2)$ and $R_1(x_1, x_2)$ are biquadratic polynomials in x_1 and x_2 given by

$$\begin{aligned} R(x_1, x_2) &= -x_1^2 x_2^2 + 6l_1 x_1 x_2 + 2lc(x_1 + x_2) + c^2 - k^2 \\ R_1(x_1, x_2) &= -6l_1 x_1^2 x_2^2 - (c^2 - k^2)(x_1 + x_2)^2 - 4lcx_1 x_2 (x_1 + x_2) \\ &\quad + 6l_1(c^2 - k^2) - 4c^2 l^2. \end{aligned}$$

The discriminant separability condition

$$\mathcal{D}_s(Q)(x_1, x_2) = P(x_1)P(x_2), \quad \mathcal{D}_{x_i}(Q)(s, x_2) = J(s)P(x_j)$$

is satisfied with polynomials

$$\begin{aligned} J(s) &= 4s^3 + (c^2 - k^2 - 3l_1^2)s - l^2 c^2 + l_1^3 - l_1 k^2 + l_1 c^2 \\ P(x_i) &= -x_i^4 + 6l_1 x_i^2 + 4lcx_i + c^2 - k^2, \quad i = 1, 2. \end{aligned}$$

Moreover, as it has been explained in [13], all the main steps of the Kowalevski integration procedure from [19] (see also [18], [17]) now follow as easy and transparent logical consequences of the theory of discriminantly separable polynomials.

There are two natural and important questions in this context:

1) *Are there any other discriminantly separable polynomials of degree two in each of three variables, beside those constructed from pencils of conics? In addition, the question is to perform a classification of such polynomials.*

2) *Are there other integrable dynamical systems related to discriminantly separable polynomials?*

The main issue of this paper is to address these two key questions.

In order to make precise the first question, one needs to specify a gauge group or the classes of equivalence up to which a classification would be performed. This leads to the group of Möbius transformations, as introduced in Corollary 3 of [13]:

$$x_1 \mapsto \frac{a_1 x_1 + b_1}{c_1 x_1 + d_1}, \quad x_2 \mapsto \frac{a_2 x_2 + b_2}{c_2 x_2 + d_2}, \quad w \mapsto \frac{a_3 w + b_3}{c_3 w + d_3}. \quad (3)$$

The family of discriminantly separable polynomials in three variables of degree two in each of them, constructed from pencils of conics served as a motivation to introduce more general classes of **discriminantly separable polynomials**. Let us recall here the definitions from [13]: a polynomial $F(x_1, \dots, x_n)$ is *discriminantly separable* if there exist polynomials $f_i(x_i)$ such that for every $i = 1, \dots, n$

$$\mathcal{D}_{x_i} F(x_1, \dots, \hat{x}_i, \dots, x_n) = \prod_{j \neq i} f_j(x_j).$$

It is *symmetrically discriminantly separable* if $f_2 = f_3 = \dots = f_n$, while it is *strongly discriminantly separable* if $f_1 = f_2 = f_3 = \dots = f_n$. It is *weakly discriminantly separable* if there exist polynomials $f_i^j(x_i)$ such that for every $i = 1, \dots, n$: $\mathcal{D}_{x_i} F(x_1, \dots, \hat{x}_i, \dots, x_n) = \prod_{j \neq i} f_j^i(x_j)$.

The classification of strongly discriminantly separable polynomials $\mathcal{F}(x_1, x_2, x_3)$ of degree two in each of three variables modulo fractional-linear transformations from formulae (3) with $a_1 = a_2 = a_3$, $b_1 = b_2 = b_3$, $c_1 = c_2 = c_3$, $d_1 = d_2 = d_3$, as the gauge group, is one of the main tasks of the present paper.

This classification is heavily based on the classification of pencils of conics (see [15], for example, for more details about pencils of conics). In the case of general position, the conics of a pencil intersect in four distinct points, and we code such situation with $(1, 1, 1, 1)$, see Fig. 1, and denote it by A. It corresponds to the case where polynomial P has four simple zeros. In this case, the family of strongly discriminantly separable polynomials coincides with the family constructed in [13] from a general pencil of conics. This family, as it has been indicated in [13], corresponds to the two-valued Buchstaber-Novikov group associated with a cubic curve

$$\Gamma_2 : y^2 = J(s).$$

Two-valued groups of the form (Γ_2, Z_2) have been introduced in [7]. The so-called Kowalevski change of variables appeared to be an infinitesimal of the two-valued operation in this group, see [13]. (The theory of n valued-groups originates from a pioneering paper [9]. For a modern account see [6], and for higher-genus analogues, see [8].)

Other cases within the classification with a nonzero polynomial P correspond to the situation where:

- B the polynomial P has two simple zeros and one double zero, we code it $(1, 1, 2)$, and the conics of the corresponding pencil intersect in two simple points, and they have a common tangent in the third point of intersection, see Fig. 2;

- C the polynomial P has two double zeros, code $(2, 2)$, and the conics of the corresponding pencil intersect in two points, having a common tangent in each of the points of intersection, see Fig. 3;
- D the polynomial P has one simple zero and one triple zero, $(1, 3)$, and the conics of the corresponding pencil intersect in one simple point, and they have another common point of tangency of third order, see Fig. 4;
- E the polynomial P has one quadruple zero, (4) , and the conics of the corresponding pencil intersect in one point, having tangency of fourth order there, see Fig. 5.

For more details, see Section 2.

Referring to the second key question, about constructing other integrable dynamical systems related to discriminantly separable polynomials, we address it in the following direction: to trace a connection between the discriminantly separable polynomials of degree two in each of three variables with the discrete integrable systems on quad-graphs. See [3] and [4] for more detail about the quad-graphs; some basic notions from there are collected in Section 3. For each discriminantly separable polynomial we construct explicit discrete integrable system in Section 3. Thus, we establish a natural correspondence between our classification of discriminantly separable polynomials of degree two in three variables, mentioned above and described in Section 2, and the classification of integrable quad-graphs performed in [4]. This correspondence gives, as a fit back benefit, a geometric interpretation of constructions of Adler, Bobenko and Suris in terms of pencils of conics. Developing further this correspondence, in the case coded $(1, 1, 1, 1)$, with a polynomial P with four distinct zeros, we come to the conclusion that such integrable quad-graphs are related to the two-valued Buchstaber-Novikov groups associated with elliptic curves, i.e. those defined on \mathbb{CP}^1 .

Recently, a new class of geometric quad-graphs, of the so-called line congruences, associated with pencils of quadrics has been introduced in [16].

2 Classification of strongly discriminantly separable polynomials of degree two in three variables

In this section we will classify strongly discriminantly separable polynomials $\mathcal{F}(x_1, x_2, x_3) \in \mathbb{C}[x_1, x_2, x_3]$ which are of degree two in each variable, modulo a gauge group of the following fractional-linear transformations

$$x_1 \mapsto \frac{ax_1 + b}{cx_1 + d}, x_2 \mapsto \frac{ax_2 + b}{cx_2 + d}, x_3 \mapsto \frac{ax_3 + b}{cx_3 + d}. \quad (4)$$

Let

$$\mathcal{F}(x_1, x_2, x_3) = \sum_{i,j,k=0}^2 a_{ijk} x_1^i x_2^j x_3^k \quad (5)$$

be a strongly discriminantly separable polynomial with

$$\mathcal{D}_{x_i} \mathcal{F}(x_j, x_k) = P(x_j)P(x_k), \quad (i, j, k) = c.p.(1, 2, 3). \quad (6)$$

Here, by $\mathcal{D}_{x_i} \mathcal{F}(x_j, x_k)$ we denote the discriminant of \mathcal{F} considered as a quadratic polynomial in x_i .

By plugging (5) into (6) for a given polynomial $P(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$, we get a system of 75 equations of degree two with 27 unknowns a_{ijk} .

Theorem 1 *The strongly discriminantly separable polynomials $\mathcal{F}(x_1, x_2, x_3)$ satisfying (6) modulo fractional linear transformations are exhausted by the following list depending on the structure of the roots of a non-zero polynomial $P(x)$:*

(A) *P has four simple zeros, with the canonical form $P(x) = (k^2x^2 - 1)(x^2 - 1)$,*

$$\begin{aligned} \mathcal{F}_A = & \frac{1}{2}(-k^2x_1^2 - k^2x_2^2 + 1 + k^2x_1^2x_2^2)x_3^2 + (1 - k^2)x_1x_2x_3 \\ & + \frac{1}{2}(x_1^2 + x_2^2 - k^2x_1^2x_2^2 - 1), \end{aligned}$$

(B) *P has two simple zeros and one double zero, with the canonical form $P(x) = x^2 - e^2$, $e \neq 0$,*

$$\mathcal{F}_B = x_1x_2x_3 + \frac{e}{2}(x_1^2 + x_2^2 + x_3^2 - e^2),$$

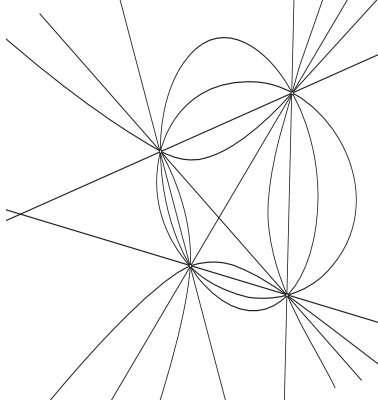


Figure 1: Pencil with four simple points

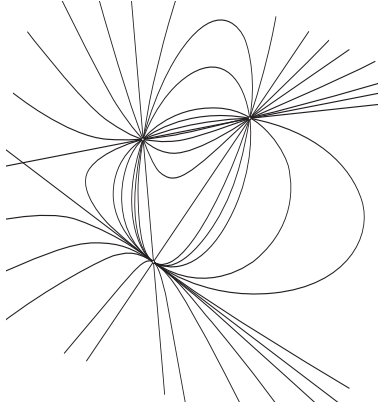


Figure 2: Pencil with one double and two simple points

(C) P has two double zeros, with the canonical form $P(x) = x^2$,

$$\begin{aligned}\mathcal{F}_{C1} &= \lambda x_1^2 x_3^2 + \mu x_1 x_2 x_3 + \nu x_2^2, & \mu^2 - 4\lambda\nu &= 1, \\ \mathcal{F}_{C2} &= \lambda x_1^2 x_2^2 x_3^2 + \mu x_1 x_2 x_3 + \nu, & \mu^2 - 4\lambda\nu &= 1,\end{aligned}$$

(D) P has one simple and one triple zero, with the canonical form $P(x) = x$,

$$\mathcal{F}_D = -\frac{1}{2}(x_1 x_2 + x_2 x_3 + x_1 x_3) + \frac{1}{4}(x_1^2 + x_2^2 + x_3^2),$$

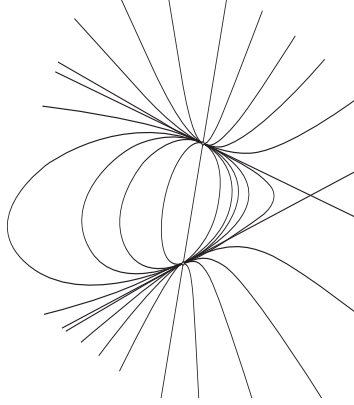


Figure 3: Pencil with two double points

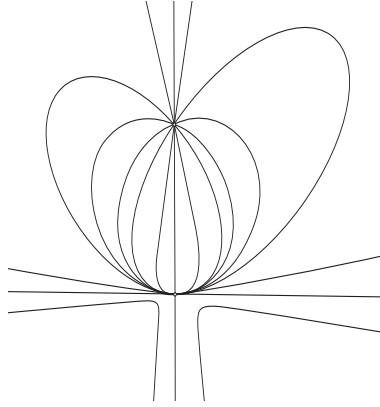


Figure 4: Pencil with one simple and one triple point

(E) P has one quadruple zero, with the canonical form $P(x) = 1$,

$$\mathcal{F}_{E1} = \lambda(x_1 + x_2 + x_3)^2 + \mu(x_1 + x_2 + x_3) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{E2} = \lambda(x_2 + x_3 - x_1)^2 + \mu(x_2 + x_3 - x_1) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{E3} = \lambda(x_1 + x_3 - x_2)^2 + \mu(x_1 + x_3 - x_2) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{E4} = \lambda(x_1 + x_2 - x_3)^2 + \mu(x_1 + x_2 - x_3) + \nu, \quad \mu^2 - 4\lambda\nu = 1.$$

The corresponding pencils of conics are of types $(1,1,1,1)$, $(1,1,2)$, $(2,2)$, $(1,3)$ and (4) in the cases (A)-(E) respectively. (These pencils are presented on Figures 1-5 respectively.)

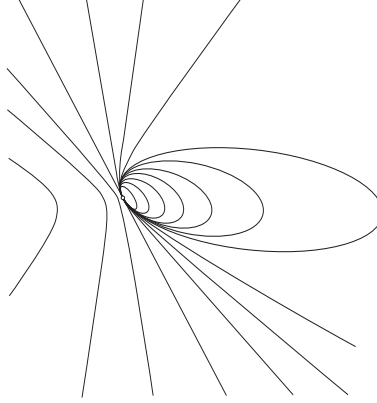


Figure 5: Pencil with one quadruple point

Proof. Proof is done by a straightforward calculation by solving the system of equations (6) for the canonical representatives of the polynomials P. For example, in the case when

$$P(x) = (k^2x^2 - 1)(x^2 - 1),$$

as the solutions of the system (6) we get the polynomials:

$$\begin{aligned} \mathcal{F}_{A1,2} &= \pm \left(\left(-\frac{k^2}{2}x_1^2 - \frac{k^2}{2}x_2^2 + \frac{1}{2} + \frac{k^2}{2}x_1^2x_2^2 \right)x_3^2 + (1 - k^2)x_1x_2x_3 \right. \\ &\quad \left. + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \frac{k^2}{2}x_1^2x_2^2 - \frac{1}{2} \right), \\ \mathcal{F}_{A2,2} &= \pm \left(\left(-\frac{k^2}{2}x_1^2 - \frac{k^2}{2}x_2^2 + \frac{1}{2} + \frac{k^2}{2}x_1^2x_2^2 \right)x_3^2 + (-1 + k^2)x_1x_2x_3 \right. \\ &\quad \left. + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \frac{k^2}{2}x_1^2x_2^2 - \frac{1}{2} \right), \\ \mathcal{F}_{A3,2} &= \pm \left(\left(\frac{k}{2} - \frac{k}{2}x_1^2 - \frac{k}{2}x_2^2 + \frac{k^3}{2}x_1^2x_2^2 \right)x_3^2 + (1 - k^2)x_1x_2x_3 \right. \\ &\quad \left. + \frac{k}{2}x_1^2 + \frac{k}{2}x_2^2 - \frac{k}{2}x_1^2x_2^2 - \frac{1}{2k} \right), \\ \mathcal{F}_{A4,2} &= \pm \left(\left(\frac{k}{2} - \frac{k}{2}x_1^2 - \frac{k}{2}x_2^2 + \frac{k^3}{2}x_1^2x_2^2 \right)x_3^2 + (-1 + k^2)x_1x_2x_3 \right. \\ &\quad \left. + \frac{k}{2}x_1^2 + \frac{k}{2}x_2^2 - \frac{k}{2}x_1^2x_2^2 - \frac{1}{2k} \right). \end{aligned}$$

The polynomials $\mathcal{F}_{A1,2}$ and $\mathcal{F}_{A2,2}$ are equivalent modulo the fractional-linear transformations (4) $x_i \mapsto -x_i$, $i = 1, 2, 3$. The same is true for the

polynomials $\mathcal{F}_{A3_{1,2}}$ and $\mathcal{F}_{A4_{1,2}}$.

Further, acting by the transformations $x_i \mapsto 1/(kx_i)$, $i = 1, 2, 3$, on $\mathcal{F}_{A3_1}(x_1, x_2, x_3)$ we get the gauge equivalence between $\mathcal{F}_{A1_{1,2}}$ and $\mathcal{F}_{A3_{1,2}}$.

In the same way we get the polynomials in the cases [B]-[D]. \square

We consider more closely the case when $P(x)$ has four simple zeros.

Proposition 1 *In the case when the polynomial $P(x)$ has four simple zeros, all strongly discriminantly separable polynomials $\mathcal{F}(x_1, x_2, x_3)$ satisfying (6) are, modulo the fractionally-linearly transformations, equivalent to*

$$\mathcal{F}(x_1, x_2, x_3) = (x_1 + x_2 + x_3)(4x_1x_2x_3 - g_3) - \left(x_1x_2 + x_1x_3 + x_2x_3 + \frac{g_2}{4}\right)^2. \quad (7)$$

Proof. The gauge transformation

$$x_i \mapsto -\frac{12k^2x_i + k^2 - 5}{12k^2x_i - 5k^2 + 1}, \quad i = 1, 2, 3,$$

transforms the polynomial $\mathcal{F}_A(x_1, x_2, x_3)$ into $54(k-1)^3(k+1)^3\hat{\mathcal{F}}_A(x_1, x_2, x_3)$, with

$$\begin{aligned} \hat{\mathcal{F}}_A(x_1, x_2, x_3) = & -6912k^8((x_1 + x_2 + x_3)(4x_1x_2x_3 - g_3) - (x_1x_2 + x_1x_3 \\ & + x_2x_3 + \frac{g_2}{4})^2). \end{aligned}$$

Here, g_2 and g_3 denote the coefficients of the Weierstrass normal form $4x^3 - g_2x - g_3$. For a polynomial $P(x) = A + 4Bx + 6Cx^2 + 4Dx^3 + Ex^4$, these coefficients are given by $g_2 = AE - 4BD + 3C^2$, $g_3 = ACE + 2BDC - AD^2 - B^2E - C^3$. \square

Notice that expanding (7) as a polynomial of x_3 , it becomes

$$\begin{aligned} \mathcal{F}(x_1, x_2, x_3) = & -(x_1 - x_2)^2x_3^2 + (2x_1^2x_2 + 2x_1x_2^2 - g_3 - x_1\frac{g_2}{2} - x_2\frac{g_2}{2})x_3 \\ & - \frac{g_2^2}{16} - x_1g_3 - x_1^2x_2^2 - x_2g_3 - \frac{g_2}{2}x_1x_2. \end{aligned} \quad (8)$$

Comparing with [7] we get

Corollary 1 *If a polynomial P has four simple zeros, all strongly discriminantly separable polynomials \mathcal{F} satisfying (6) are equivalent to the two-valued group*

$$(x_1x_2 + x_2x_3 + x_1x_3)(4 + g_3x_1x_2x_3) = \left(x_1 + x_2 + x_3 - \frac{g_2x_1x_2x_3}{4}\right)^2.$$

3 From discriminantly separable polynomials to integrable quad-graphs

Now, we will establish a connection between the discriminantly separable polynomials and the theory of integrable systems on quad graphs. Latter has been developed by Adler, Bobenko, Suris [3],[4], see also [1], [2]. Moreover, we are going to construct an integrable quad graph associated to an arbitrary given discriminantly separable polynomial.

Recall that the basic building blocks of systems on the quad-graphs are the equations on quadrilaterals of the form

$$Q(x_1, x_2, x_3, x_4) = 0 \tag{9}$$

where Q is a polynomial of degree one in each argument, i.e. Q is a multi-affine polynomial. The field variables x_i are assigned to four vertices of a quadrilateral as in a Figure 6. Besides depending on the variables $x_1, \dots, x_4 \in \mathbb{C}$, the polynomial Q still depends on two parameters $\alpha, \beta \in \mathbb{C}$ that are assigned to the edges of a quadrilateral. It is assumed that opposite edges carry the same parameter.

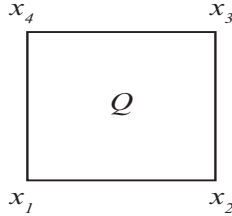


Figure 6: An elementary quadrilateral. A quad-equation $Q(x_1, x_2, x_3, x_4) = 0$.

Equations of type (9) are called *the quad-equations*. The equation (9) can be solved for each variable, and the solution is a rational function of the

other three variables. A solution (x_1, x_2, x_3, x_4) of equation (9) is *singular* with respect to x_i if it also satisfies the equation $Q_{x_i}(x_1, x_2, x_3, x_4) = 0$.

Following [4] we consider the idea of integrability as *a consistency*, see Figure 7. We assign six quad-equations to the faces of the coordinate cube. The system is said to be *3D-consistent* if the three values for x_{123} obtained from the equations on the right, back and top faces coincide for arbitrary initial data x, x_1, x_2, x_3 .

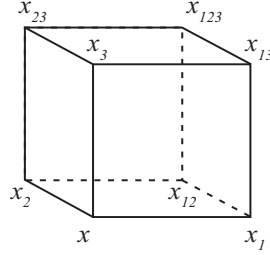


Figure 7: A 3D consistency.

Then, by applying the discriminant-like operators introduced in [4]

$$\delta_{x,y}(Q) = Q_x Q_y - Q Q_{xy}, \quad \delta_x(h) = h_x^2 - 2h h_{xx}, \quad (10)$$

one can make descent from the faces to the edges and then to the vertices of the cube: from a multiaffine polynomial $Q(x_1, x_2, x_3, x_4)$ to a biquadratic polynomial $h(x_i, x_j) := \delta_{x_k, x_l}(Q(x_i, x_j, x_k, x_l))$ and further to a polynomial $P(x_i) = \delta_{x_j}(h(x_i, x_j))$ of degree up to four. By using of relative invariants of polynomials under fractional linear transformations, the formulae that express Q through the biquadratic polynomials of three edges, were derived in [4]:

$$\frac{2Q_{x_1}}{Q} = \frac{h_{x_1}^{12} h^{34} - h_{x_1}^{14} h^{23} + h^{23} h_{x_3}^{34} - h_{x_3}^{23} h^{34}}{h^{12} h^{34} - h^{14} h^{23}}. \quad (11)$$

A biquadratic polynomial $h(x, y)$ is said to be *nondegenerate* if no polynomial in its equivalence class with respect to the fractional linear transformations, is divisible by a factor of the form $x - c$ or $y - c$, with $c = \text{const}$. A multiaffine function $Q(x_1, x_2, x_3, x_4)$ is said to be of *type Q* if all four of its accompanying biquadratic polynomials h^{jk} are nondegenerate. Otherwise, it is of *type H*. Previous notions were introduced in [4].

Lemma 1 *Given a biquadratic polynomial*

$$h(x_1, x_2) = h_{22}x_1^2x_2^2 + h_{21}x_1^2x_2 + h_{12}x_1x_2^2 + h_{20}x_1^2 + h_{02}x_2^2 + h_{11}x_1x_2 + h_{10}x_1 + h_{01}x_2 + h_{00} \quad (12)$$

satisfying the condition:

$$\delta_{x_1}(h) = P(x_2), \quad \delta_{x_2}(h) = P(x_1) \quad (13)$$

where the discriminant $\delta_{x_i}(h)$ is given by (10) and $P(x_i)$ is a nonzero polynomial of degree up to four in the canonical form. Then, up to the fractional linear transformations, h is equivalent to:

(A) *for $P_A(x) = (k^2x^2 - 1)(x^2 - 1)$,*

$$h = -\frac{k^2}{4h_{20}}x_1^2x_2^2 + h_{20}(x_1^2 + x_2^2) - \frac{1}{4h_{20}} \pm \frac{\sqrt{(2h_{20} - 1)(2h_{20} + 1)(2h_{20} - k)(2h_{20} + k)}}{2h_{20}}x_1x_2, \quad (14)$$

(B) *for $P_B(x) = x^2 - e^2$, $e \neq 0$,*

$$h = h_{20}(x_1^2 + x_2^2) \pm \sqrt{1 + 4h_{20}^2}x_1x_2 + \frac{e^2}{4h_{20}}, \quad (15)$$

(C) *for $P_C(x) = x^2$,*

$$\begin{aligned} h &= h_{20}x_1^2 + h_{11}x_1x_2 + h_{02}x_2^2, & h_{11}^2 - 4h_{20}h_{02} &= 1, \\ h &= h_{22}x_1^2x_2^2 + h_{11}x_1x_2 + h_{00}, & h_{11}^2 - 4h_{22}h_{00} &= 1, \end{aligned} \quad (16)$$

(D) *for $P_D(x) = x$,*

$$h = -\frac{h_{11}}{2}(x_1 - x_2)^2 + \frac{1}{4h_{11}}(x_1 + x_2) - \frac{1}{32h_{11}^3}, \quad (17)$$

(E) *for $P_E(x) = 1$,*

$$h = h_{20}(x_1 \pm x_2)^2 + h_{10}(x_1 \pm x_2) + h_{00}, \quad h_{10}^2 - 4h_{20}h_{00} = 1. \quad (18)$$

Proof. Proof is done by a straightforward calculation by solving the system of equations (13). \square

We can notice that the families of polynomials h of two variables x_1, x_2 obtained in the previous Lemma depend on the additional parameter. For example, in the case (A), they depend on the additional free parameter h_{20} . Correlation with strongly discriminantly separable polynomials $\mathcal{F}(x_1, x_2, x_3)$ is achieved through the additional assumption that coefficients of polynomials h_{ij} are quadratic functions of the parameter α .

Now, we start with an arbitrary strongly discriminantly separable polynomial

$$\mathcal{F}(x_1, x_2, \alpha)$$

of degree two in each of three variables. We are going to construct an integrable quad-graph which corresponds to $\mathcal{F}(x_1, x_2, \alpha)$. In order to define such an integrable quad-graph we have to provide corresponding biquadratic polynomial $h = h(x_1, x_2)$ and a multiaffine polynomial $Q = Q(x_1, x_2, x_3, x_4)$.

The requirement that the discriminants of $h(x_1, x_2)$ do not depend on α , see [3], [4], will be satisfied if as a biquadratic polynomials $h(x_1, x_2)$ we take

$$\hat{h}(x_1, x_2) := \frac{\mathcal{F}(x_1, x_2, \alpha)}{\sqrt{P(\alpha)}}.$$

Proposition 2 *The polynomials*

$$\hat{h}(x_1, x_2) = \frac{\mathcal{F}(x_1, x_2, \alpha)}{\sqrt{P(\alpha)}} \quad (19)$$

satisfy

$$\delta_{x_1}(\hat{h}) = P(x_2), \quad \delta_{x_2}(\hat{h}) = P(x_1).$$

The list of biquadratic polynomials $\hat{h}(x_1, x_2)$ having $P_I(x)$, $I = A, B, C, D, E$ as their discriminants, is:

(A) *for $P_A(x) = (k^2x^2 - 1)(x^2 - 1)$,*

$$\begin{aligned} \hat{h}_A(x_1, x_2) = & \frac{(-k^2x_1^2 - k^2x_2^2 + 1 + k^2x_1^2x_2^2)\alpha^2 + (1 - k^2)x_1x_2\alpha}{2\sqrt{(k^2\alpha^2 - 1)(\alpha^2 - 1)}} \\ & + \frac{x_1^2 + x_2^2 - k^2x_1^2x_2^2 - 1}{4\sqrt{(k^2\alpha^2 - 1)(\alpha^2 - 1)}}. \end{aligned}$$

Remark here that $\hat{h}_A(x_1, x_2) = h$ from (14) for

$$h_{20} = \frac{-\sqrt{k^2\alpha^2 - 1}}{2\sqrt{\alpha^2 - 1}}.$$

(B) for $P_B(x) = x^2 - e^2$, $e \neq 0$,

$$\hat{h}_B(x_1, x_2) = \frac{x_1 x_2 \alpha + \frac{e}{2}(x_1^2 + x_2^2 + \alpha^2 - e^2)}{\sqrt{\alpha^2 - e^2}},$$

and $\hat{h}_B(x_1, x_2) = h$ from (15) for

$$h_{20} = \frac{e}{2\sqrt{\alpha^2 - e^2}}.$$

(C) for $P_C(x) = x^2$,

$$\hat{h}_{C1}(x_1, x_2) = \frac{\lambda x_1^2 \alpha^2 + \mu x_1 x_2 \alpha + \nu x_2^2}{\alpha}, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\hat{h}_{C2}(x_1, x_2) = \frac{\lambda x_1^2 x_2^2 \alpha^2 + \mu x_1 x_2 \alpha + \nu}{\alpha}, \quad \mu^2 - 4\lambda\nu = 1.$$

(D) for $P_D(x) = x$,

$$\hat{h}_D(x_1, x_2) = \frac{-(x_1 x_2 + x_1 \alpha + x_2 \alpha) + \frac{1}{4}(x_1^2 + x_2^2 + \alpha^2)}{2\sqrt{\alpha}},$$

and $\hat{h}_D(x_1, x_2) = h$ from (17) for

$$h_{11} = \frac{-1}{2\sqrt{\alpha}}.$$

(E) for $P_E(x) = 1$,

$$\hat{h}_{E1}(x_1, x_2) = \lambda(x_1 + x_2 + \alpha)^2 + \mu(x_1 + x_2 + \alpha) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\hat{h}_{E2}(x_1, x_2) = \lambda(-x_1 + x_2 + \alpha)^2 + \mu(-x_1 + x_2 + \alpha) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\hat{h}_{E3}(x_1, x_2) = \lambda(x_1 - x_2 + \alpha)^2 + \mu(x_1 - x_2 + \alpha) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\hat{h}_{E4}(x_1, x_2) = \lambda(x_1 + x_2 - \alpha)^2 + \mu(x_1 + x_2 - \alpha) + \nu, \quad \mu^2 - 4\lambda\nu = 1.$$

Notice that polynomials \hat{h}_{C1} , \hat{h}_{C2} , \hat{h}_{Ei} may be rewritten as (16) and (18):

$$\hat{h}_{C1}(x_1, x_2) = \lambda \alpha x_1^2 + \mu x_1 x_2 + \frac{\nu}{\alpha} x_2^2 = \lambda' x_1^2 + \mu x_1 x_2 + \nu' x_2^2, \quad \mu^2 - 4\lambda'\nu' = 1$$

$$\hat{h}_{C2}(x_1, x_2) = \lambda \alpha x_1^2 + \mu x_1 x_2 + \frac{\nu}{\alpha} x_2^2 = \lambda' x_1^2 + \mu x_1 x_2 + \nu' x_2^2, \quad \mu^2 - 4\lambda'\nu' = 1$$

$$\begin{aligned} \hat{h}_{E1}(x_1, x_2) &= \lambda(x_1 + x_2)^2 + 2\alpha\lambda(x_1 + x_2) + \lambda\alpha^2 + \mu(x_1 + x_2) + \mu\alpha + \nu \\ &= \lambda(x_1 + x_2)^2 + (\mu + 2\alpha\lambda)(x_1 + x_2) + \lambda\alpha^2 + \mu\alpha + \nu \\ &= \lambda(x_1 + x_2)^2 + \mu'(x_1 + x_2) + \nu', \quad \mu'^2 - 4\lambda\nu' = \mu^2 - 4\lambda\nu = 1, \end{aligned}$$

and, in the same way for $\hat{h}_{E2} - \hat{h}_{E4}$.

The final problem of the reconstruction of the *quad-equations of type Q* which correspond to the strongly discriminantly separable polynomials obtained in Theorem 1 is done with a use of the formulae (11) and with the polynomials h^{ij} replaced by \hat{h}^{ij} .

Here we take $\hat{h}^{12} = \hat{h}_I(x_1, x_2; \alpha)$, $\hat{h}^{23} = \hat{h}_I(x_2, x_3; \beta)$, $\hat{h}^{34} = \hat{h}_I(x_3, x_4; \alpha)$, $\hat{h}^{14} = \hat{h}_I(x_1, x_4; \beta)$, for $I = A, B, C, D, E$.

Theorem 2 *The quad-equations of type Q that correspond to the biquadratic polynomials*

$$\hat{h}(x_1, x_2; \alpha) = \frac{\mathcal{F}(x_1, x_2, \alpha)}{\sqrt{P_I(\alpha)}}$$

are given in the following list:

$$\begin{aligned} \hat{Q}_A &= \frac{\beta\sqrt{P_A(\alpha)} + \alpha\sqrt{P_A(\beta)}}{k^2\alpha^2\beta^2 - 1} \sqrt{\frac{\alpha^2 - 1}{k^2\alpha^2 - 1}} \sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}} (k^2x_1x_2x_3x_4 + 1) \\ &\quad + \sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}} (x_1x_2 + x_3x_4) + \sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}} (x_1x_4 + x_2x_3) \\ &\quad + \frac{\beta\sqrt{P_A(\alpha)} + \alpha\sqrt{P_A(\beta)}}{k^2\alpha^2\beta^2 - 1} (x_1x_3 + x_2x_4) = 0, \\ \hat{Q}_B &= \sqrt{\alpha^2 - e^2} (x_1x_2 + x_3x_4) + \sqrt{\beta^2 - e^2} (x_1x_4 + x_2x_3) \\ &\quad + \frac{\alpha\sqrt{\beta^2 - e^2} + \beta\sqrt{\alpha^2 - e^2}}{e} (x_1x_3 + x_2x_4) \\ &\quad - \frac{\sqrt{\beta^2 - e^2}\sqrt{\alpha^2 - e^2}(\alpha\sqrt{\beta^2 - e^2} + \beta\sqrt{\alpha^2 - e^2})}{e} = 0, \\ \hat{Q}_C &= (\alpha - \frac{1}{\alpha})(x_1x_2 + x_3x_4) + (\beta - \frac{1}{\beta})(x_1x_4 + x_2x_3) \\ &\quad - (\alpha\beta - \frac{1}{\alpha\beta})(x_1x_3 + x_2x_4) = 0, \\ \hat{Q}_D &= \sqrt{\alpha}(x_1 - x_4)(x_2 - x_3) + \sqrt{\beta}(x_1 - x_2)(x_4 - x_3) \\ &\quad - \sqrt{\alpha}\sqrt{\beta}(\sqrt{\alpha} + \sqrt{\beta})(x_1 + x_2 + x_3 + x_4) \\ &\quad + \sqrt{\alpha}\sqrt{\beta}(\sqrt{\alpha} + \sqrt{\beta})(\alpha + \sqrt{\alpha\beta} + \beta) = 0, \\ \hat{Q}_E &= \alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3) - \alpha\beta(\alpha + \beta) = 0. \end{aligned}$$

Proof. From (11) we get that when

$$\begin{aligned}\hat{h}^{12} &= \hat{h}_A(x_1, x_2; \alpha), \quad \hat{h}^{23} = \hat{h}_A(x_2, x_3; \beta), \\ \hat{h}^{34} &= \hat{h}_A(x_3, x_4; \alpha), \quad \hat{h}^{14} = \hat{h}_A(x_1, x_4; \beta),\end{aligned}$$

corresponding Q is

$$\hat{Q}_A = a_1 x_1 x_2 x_3 x_4 + a_6 (x_1 x_2 + x_3 x_4) + a_8 (x_1 x_4 + x_2 x_3) + a_{10} (x_1 x_3 + x_2 x_4) + a_{16}$$

with

$$\begin{aligned}a_1 &= \frac{a_{10} k^2 \sqrt{P_A(\alpha)} \sqrt{P_A(\beta)}}{k^2 \alpha^2 \beta^2} \\ a_6 &= \frac{a_{10} \sqrt{P_A(\alpha)} (k^2 \alpha^2 \beta^2 - 1)}{(\alpha \sqrt{P_A(\beta)} + \beta \sqrt{P_A(\alpha)}) (k^2 \alpha^2 - 1)} \\ a_8 &= \frac{a_{10} \sqrt{\beta^2 - 1} (\beta \sqrt{P_A(\alpha)} - \alpha \sqrt{P_A(\beta)})}{\sqrt{k^2 \beta^2 - 1} (\alpha^2 - \beta^2)} \\ a_{16} &= \frac{a_{10} \sqrt{P_A(\alpha)} \sqrt{P_A(\beta)}}{k^2 \alpha^2 \beta^2}.\end{aligned}$$

Recall here that $P_A(x) = (k^2 x^2 - 1)(x^2 - 1)$. Choosing

$$a_{10} = \frac{\beta \sqrt{P_A(\alpha)} + \alpha \sqrt{P_A(\beta)}}{k^2 \alpha^2 \beta^2 - 1}$$

we finally get \hat{Q}_A .

In the same way for the other choices $\hat{h}^{ij} = \hat{h}_I(x_i, x_j)$, $I = B, C, D, E$ from (11) we get the expressions for $\hat{Q}_B - \hat{Q}_E$. □

Remark 1 *The list of biquadratic polynomials from Lemma 1 contains the list of h 's obtained in [4] as particular cases. More precisely, by plugging*

$$h_{20} = \frac{1}{2\alpha}$$

into (14) it becomes

$$h = \frac{1}{2\alpha} (k^2 \alpha^2 x_1^2 x_2^2 + 2 \sqrt{P_A(\alpha)} x_1 x_2 - x_1^2 - x_2^2 + \alpha^2). \quad (20)$$

Next, with a choice

$$h_{20} = \frac{\alpha}{1 - \alpha^2}$$

(15) turns into

$$h = \frac{\alpha}{1 - \alpha^2}(x_1^2 + x_2^2) - \frac{1 + \alpha^2}{1 - \alpha^2}x_1x_2 + \frac{e^2(1 - \alpha^2)}{4\alpha}. \quad (21)$$

Finally, for

$$h_{11} = -\frac{1}{2\alpha}$$

from (17) we get

$$h = \frac{1}{4\alpha}(x_1 - x_2)^2 - \frac{\alpha}{2}(x_1 + x_2) + \frac{\alpha^3}{4}. \quad (22)$$

The expressions for h in [4] for $P(x) = x^2$ and $P(x) = 1$ are the same as the corresponding h in Lemma 1.

In the same manner, the list obtained in Theorem 2 contains the list of the multiaffine equations of type Q obtained in [4]:

$$\begin{aligned} & sn(\alpha)sn(\beta)sn(\alpha + \beta)(k^2x_1x_2x_3x_4 + 1) - sn(\alpha)(x_1x_2 + x_3x_4) \\ & - sn(\beta)(x_1x_4 + x_2x_3) + sn(\alpha + \beta)(x_1x_3 + x_2x_4) = 0, \end{aligned} \quad (23)$$

$$\begin{aligned} & (\alpha - \alpha^{-1})(x_1x_2 + x_3x_4) + (\beta - \beta^{-1})(x_1x_4 + x_2x_3) - (\alpha\beta - \alpha^{-1}\beta^{-1})(x_1x_3 + x_2x_4) \\ & + \frac{\delta}{4}(\alpha - \alpha^{-1})(\beta - \beta^{-1})(\alpha\beta - \alpha^{-1}\beta^{-1}) = 0, \end{aligned} \quad (24)$$

$$\begin{aligned} & \alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3) - \alpha\beta(\alpha + \beta)(x_1 + x_2 + x_3 + x_4) \\ & + \alpha\beta(\alpha + \beta)(\alpha^2 + \alpha\beta + \beta^2) = 0, \end{aligned} \quad (25)$$

$$\alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3) - \delta\alpha\beta(\alpha + \beta) = 0. \quad (26)$$

In order to see the correspondence between $\hat{Q}_A = 0$ and (23) we need to go back to the system 6. In the process of solving the systems under the current assumptions, a step before the final solution, one comes to the following relation:

$$\alpha\beta\gamma(k^2x_1x_2x_3x_4 + 1) + \alpha(x_1x_2 + x_3x_4) + \beta(x_1x_4 + x_2x_3) + \gamma(x_1x_3 + x_2x_4) = 0 \quad (27)$$

where

$$\gamma = \frac{\alpha\sqrt{P_A(\beta)} + \beta\sqrt{P_A(\alpha)}}{k^2\alpha^2\beta^2 - 1}.$$

Then, (23) is obtained from (27) with $\alpha \mapsto \text{sn}(\alpha)$, $\sqrt{P_A(\alpha)} \mapsto \text{sn}'(\alpha)$ and similarly for β . If we replace the parameters α, β in (23) by

$$\alpha \mapsto \sqrt{\frac{\alpha^2 - 1}{k^2\alpha^2 - 1}}, \quad \beta \mapsto \sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}}$$

then the correspondence of \hat{Q}_A and (27) is obvious.

The correspondence of \hat{Q}_B and (24) with $\delta = e^2$ is achieved with a change

$$\alpha \mapsto \frac{\alpha - \sqrt{\alpha^2 - e^2}}{e}, \quad \beta \mapsto \frac{\beta - \sqrt{\beta^2 - e^2}}{e}.$$

Notice here that with the last change we get

$$\alpha - \alpha^{-1} \mapsto \frac{-2\sqrt{\alpha_1^2 - e^2}}{e}$$

and similarly for β .

Finally, a change of parameters

$$\alpha \mapsto \sqrt{\alpha}, \quad \beta \mapsto \sqrt{\beta}$$

brings $\hat{Q}_D = 0$ into the form of the corresponding (25).

The equation $Q(x_1, x_2, x_3, x_4) = 0$ that corresponds to the polynomial \mathcal{F} given with (7) is obtained in [3]:

$$\begin{aligned} & a_0 x_1 x_2 x_3 x_4 + a_1 (x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4) + a_2 (x_1 x_3 + x_2 x_4) \\ & + \bar{a}_2 (x_1 x_2 + x_3 x_4) + \tilde{a}_2 (x_1 x_4 + x_2 x_3) + a_3 (x_1 + x_2 + x_3 + x_4) + a_4 = 0, \end{aligned}$$

where the coefficients are

$$a_0 = a + b, \quad a_1 = -\beta a - \alpha b, \quad a_2 = \beta^2 a + \alpha^2 b,$$

$$\bar{a}_2 = \frac{ab(a+b)}{2(\alpha-\beta)} + \beta^2 a - (2\alpha^2 - \frac{g_2}{4})b,$$

$$\tilde{a}_2 = \frac{ab(a+b)}{2(\beta-\alpha)} + \alpha^2 b - (2\beta^2 - \frac{g_2}{4})a,$$

$$a_3 = \frac{g_3}{2}a_0 - \frac{g_2}{4}a_1, \quad a_4 = \frac{g_2^2}{16}a_0 - g_3a_1,$$

$$\text{and } a^2 = 4\alpha^3 - g_2\alpha - g_3, \quad b^2 = 4\beta^3 - g_2\beta - g_3.$$

Previous calculations establish a relation between strongly discriminantly separable polynomials in three variables of degree two in each and biquadratic polynomials from the theory of quad-graphs which correspond to *the diagonal elements* of table on page 11, paper [4]. Remaining three *off-diagonal cases* of that table from [4], p.11, appear to be connected with the symmetrically discriminantly separable polynomials (see [13] and the Introduction for the definition of the symmetrically discriminantly separable polynomials). The next Remark is devoted to that case.

Remark 2 *The three remaining, off-diagonal, biquadratic polynomials obtained in the list in [4], p.11, are:*

$$\begin{aligned} h_1(x_1, x_2) &= \alpha x_2^2 \pm x_1 x_2 + \frac{1}{4\alpha}, \\ h_2(x_1, x_2) &= \pm \frac{1}{4}(x_2 - \alpha)^2 \mp x_1, \\ h_3(x_1, x_2) &= \lambda x_2^2 + \mu x_2 + \nu, \quad \mu^2 - 4\lambda\nu = 1, \end{aligned}$$

with the corresponding discriminants $\delta_{x_i}(h_j)$, $i = 1, 2$, $j = 1, 2, 3$:

$$\begin{aligned} \delta_{x_1}(h_1) &= x_2^2, \quad \delta_{x_2}(h_1) = x_1^2 - 1, \\ \delta_{x_1}(h_2) &= 1, \quad \delta_{x_2}(h_2) = x_1, \\ \delta_{x_1}(h_3) &= 0, \quad \delta_{x_2}(h_3) = 1. \end{aligned}$$

The polynomials h_1, h_2, h_3 obviously do not correspond to strongly discriminantly separable polynomials. However, there are symmetrically discriminantly separable polynomials which correspond to these cases. (The definition of the symmetrically discriminantly separable polynomials is given in the Introduction.) With the correlation (19), we get for the case of h_1 :

$$\mathcal{F}_1(x_1, x_2, x_3) = x_2^2 x_3^2 \pm x_1 x_2 x_3 + \frac{1}{4}$$

with

$$\mathcal{D}_{x_1}(\mathcal{F}_1) = x_2^2 x_3^2, \quad \mathcal{D}_{x_2}(\mathcal{F}_1) = (x_1^2 - 1)x_3^2, \quad \mathcal{D}_{x_3}(\mathcal{F}_1) = (x_1^2 - 1)x_2^2.$$

For the case h_2 , let us consider the next polynomial:

$$\mathcal{F}_2(x_1, x_2, x_3) = \pm \frac{(x_2 - x_3)^2}{4} \mp x_1$$

with $\mathcal{D}_{x_1}(\mathcal{F}_2) = 1$, $\mathcal{D}_{x_2}(\mathcal{F}_2) = x_1$, $\mathcal{D}_{x_3}(\mathcal{F}_2) = x_1$.

Finally, for the case h_3 we take:

$$\mathcal{F}_3(x_1, x_2, x_3) = \lambda x_3^2 + \mu x_3 + \nu, \quad \mu^2 - 4\lambda\nu = 1$$

with $\mathcal{D}_{x_1}(\mathcal{F}_3) = 0$, $\mathcal{D}_{x_2}(\mathcal{F}_3) = 0$, $\mathcal{D}_{x_3}(\mathcal{F}_3) = 1$.

Let us underline that the correspondences established in the current Section, provide a two way-link between the strongly discriminantly separable polynomials and the quad-graphs. As a benefit, we get a geometric interpretation of the biquadratic polynomials h from the theory of quad-graphs by relating them to the pencils of conics. (The pencils of conics appeared in the classification of the Yang-Baxter maps in [5].) Moreover, in the case [A], coded $(1, 1, 1, 1)$, when a polynomial P has four distinct zeros, the biquadratic polynomials of quad-graphs correspond to the two-valued Buchstaber-Novikov groups associated with elliptic curves, i.e. those defined on \mathbb{CP}^1 .

Acknowledgements

The authors would like to thank to Prof. Yu. B. Suris for his kind advices and very instructive remarks. The research was partially supported by the Serbian Ministry of Science and Technological Development, Project 174020 *Geometry and Topology of Manifolds, Classical Mechanics and Integrable Dynamical Systems*.

References

- [1] A. I. Bobenko, Yu. B. Suris, *Integrable noncommutative equations on quad-graphs. The consistency approach*. Lett. Math. Phys. 61 (2002), no. 3, 241254.
- [2] A. I. Bobenko, Yu. B. Suris, *Integrable systems on quad-graphs*. Int. Math. Res. Not. 2002, no. 11, 573611.

- [3] V. E. Adler, A. I. Bobenko, Y. B. Suris, *Classification of integrable equations on quad-graphs. The consistency approach*, Commun. Math. Phys. **233** (2003) 513-43
- [4] V. E. Adler, A. I. Bobenko, Y. B. Suris, *Discrete nonlinear hiperbolic equations. Classification of integrable cases*, Funct. Anal. Appl **43** (2009) 3-21
- [5] V. E. Adler, A. I. Bobenko, Yu. B. Suris, *Geometry of Yang-Baxter maps: pencils of conics and quadrirational mappings*. Comm. Anal. Geom. 12 (2004), no. 5, 967-1007
- [6] V. Buchstaber, *n-valued groups: theory and applications*, Moscow Mathematical Journal, **6** (2006) 57-84
- [7] V. M. Buchstaber, *Functional equations, associated with addition theorems for elliptic functions, and two-valued algebraic groups*, Russian Math. Surv. vol 45, (1990) 213-214
- [8] V. M. Buchstaber, V. Dragović, *Two-valued groups, Kummer varieties and integrable billiards*, arXiv: 1011.2716
- [9] V. M. Buchstaber, S. P. Novikov, *Formal groups, power systems and Adams operators*, Mat. Sb. (N. S) **84 (126)** (1971) 81-118 (in Russian)
- [10] V. M. Buchstaber, A. P. Veselov, *Integrable correspondences and algebraic representations of multivalued groups*, Internat. Math. Res. Notices, (1996) 381-400
- [11] G. Darboux, *Principes de géométrie analytique*, Gauthier-Villars, Paris (1917) 519 p.
- [12] V. Dragović, *Poncelet-Darboux curves, their complete decomposition and Marden theorem* Int. Math. Res. Notes (2011) no. 15, 35023523
- [13] V. Dragović, *Generalization and geometrization of the Kowalevski top*, Communications in Math. Phys. 298 (2010), no. 1, 37-64
- [14] V. Dragović, K. Kukić, *New examples of systems of the Kowalevski type*, Regular and Chaotic Dynamics, Vol. 16, No. 5, 484-495 (2011)
- [15] V. Dragović, M. Radnović, *Poncelet porisms and beyond* Springer 2011
- [16] V. Dragović, M. Radnović, *Billiard algebra, integrable line congruences and DR-nets* J. Nonlinear Mathematical Physics, Vol. 19, no. 3, 2012

- [17] V. V. Golubev, *Lectures on the integration of motion of a heavy rigid body around a fixed point*, Gostechizdat, Moscow, 1953 [in Russian], English translation: Israel program for scientific literature, 1960.
- [18] F. Kotter, *Sur le cas traite par M-me Kowalevski de rotation d'un corps solide autour d'un point fixe*, Acta Math. **17** (1893)
- [19] S. Kowalevski, *Sur la probleme de la rotation d'un corps solide autour d'un point fixe*, Acta Math. **12** (1889) 177-232